Graph Puzzles III.2-preview: Strong five-cycle double covers

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Abstract: We prove that a circuit C of a cubic graph G is a member of a five-cycle double cover whenever $G - V(C)$ is 3-edge-colourable. The reverse implication is known. The proof uses the bilinear balancing lemma from the proof of Sabidussi's compatibility conjecture. It follows that every circuit in a critical or permutation snark is a member of a five-cycle double cover. We also prove the strong five-cycle double cover property when G has an induced hexagon H and $G - E(H)$ has a proper 3-edge-colouring whose spoke colours, in cyclic order, are $1, 1, 2, 2, 3, 3$. The characterisation of snarks of colouring defect three by such a hexagon gives this property for all these snarks.

1 Introduction

A *circuit* is a nonempty connected 2-regular subgraph. A *cycle* is an even subgraph, identified with its edge set. A *k-cycle double cover* is an indexed family of at most k cycles in which every edge occurs twice. Empty members may be adjoined, and repeated members retain their multiplicities. In a cubic graph, each cycle is a union of vertex-disjoint circuits.

We say that a cubic graph has the *strong k-cycle double cover property* if every circuit is a component of a member of some k-cycle double cover. We follow the convention of Hoffmann-Ostenhof [HO13][1]. Requiring the circuit itself to be a member is stronger than requiring it to be a component of a member.

Throughout the paper graphs are finite and loopless, with parallel edges allowed. The contraction G/C retains the edges outside C, including loops created by contraction. Our first result characterises the circuits that are members of five-cycle double covers.

[1] Arthur Hoffmann-Ostenhof. *A note on 5-cycle double covers*. Graphs and Combinatorics 29(4) (2013), 977–979. doi:10.1007/s00373-012-1169-8.

Theorem 1.1 (Prescribed circuit)**.** *Let G be a connected cubic graph and let C be a circuit of G. The following conditions are equivalent.*

(i) *G has a five-cycle double cover with C as a member.*

(ii) *$G - V(C)$ has a proper 3-edge-colouring.*

(iii) *$G - E(C)$ has a proper 3-edge-colouring.*

(iv) *G/C has a nowhere-zero $\mathbb{F}_2^2$-flow.*

Li, Hao, Luo and Zhang [LHLZ25][2], Lemma 5.8, prove (i)⇒(iv) for every cycle C, without assuming that $G - E(C)$ is connected. Hoffmann-Ostenhof, Zhang and Zhang [HOZZ19][3], Theorem 3.1, proved (iv)⇒(i) for a non-separating cycle, meaning that $G - E(C)$ is connected. Li et al. also treat non-separating cycles under bounds on the rank of the cycle space of $G-E(C)$ and the number of components of C; see [LHLZ25, Theorem 1.6]. We remove the connectedness assumption in the sufficiency direction for circuits in cubic graphs.

The final step uses the construction from two even subgraphs in Liu, Hao, Luo and Zhang [LHLZ23][4], Theorem 1.6. To satisfy its hypotheses, we permute the three edge colours independently in the components of $G-E(C)$. The parity lemma in section 2 chooses these permutations so that the resulting flow on G has an even number of endpoints of zero-valued edges in each component of $G - E(C)$. A T-join then gives the second cycle needed for the construction.

A cubic graph G is *critical* if $G - \{u, v\}$ is 3-edge-colourable for every edge uv. A *cubic permutation graph* has a spanning 2-factor $A \cup B$ consisting of two induced circuits. Its remaining edges form a perfect matching between A and B. We use *snark* for a connected bridgeless cubic graph of chromatic index four. A *critical snark* is a snark satisfying the first condition; a *permutation snark* is a snark satisfying the second. The results also apply under the customary additional girth and connectivity conditions.

Theorem 1.2 (Critical and permutation graphs)**.** *Every connected critical cubic graph and every cubic permutation graph has a five-cycle double cover with any prescribed circuit as a member. In particular, critical snarks and permutation snarks have the strong five-cycle double cover property.*

The next theorem concerns a single induced hexagon and its six incident edges. It will give the result for snarks of colouring defect three.

Theorem 1.3 (An induced hexagon)**.** *Let G be a connected cubic graph with an induced hexagon $H = v_0v_1 \dots v_5v_0$. Suppose that $G - E(H)$ has a proper 3-edge-colouring in which the spokes at $v_0, \dots, v_5$ have colours $1, 1, 2, 2, 3, 3$, respectively. Then every circuit of G is a component of a member of a five-cycle double cover.*

[2] Xiaoyu Li, Rong-Xia Hao, Rong Luo and Cun-Quan Zhang. *Non-separating cycles and 5-cycle double covers.* Discrete Mathematics 348(9) (2025), 114515. doi:10.1016/j.disc.2025.114515.

[3] Arthur Hoffmann-Ostenhof, Cun-Quan Zhang and Zhang Zhang. *Cycle double covers and non-separating cycles.* European Journal of Combinatorics 81 (2019), 276–284. doi:10.1016/j.ejc.2019.06.006.

[4] Siyan Liu, Rong-Xia Hao, Rong Luo and Cun-Quan Zhang. *5-cycle double covers, 4-flows, and Catlin reduction.* SIAM Journal on Discrete Mathematics 37(1) (2023), 253–267. doi:10.1137/22M1472425.

The *colouring defect* $\mathrm{df}(G)$ is the minimum number of edges uncovered by a triple of perfect matchings. The characterisation of Karabáš, Máčajová, Nedela and Škoviera [KMNS24][5], Theorem 3.3, supplies the hexagon and edge-colouring required by theorem 1.3 when $\mathrm{df}(G) = 3$.

Corollary 1.4. *Every snark of colouring defect three has the strong five-cycle double cover property.*

[5] Ján Karabáš, Edita Máčajová, Roman Nedela and Martin Škoviera. *Cubic graphs with colouring defect 3*. Electronic Journal of Combinatorics 31(2) (2024), P2.6. doi:10.37236/12333.

1.1 Formalization

All of the results have also been formalized in Lean. The formalization is available in the author's GitHub repository: https://github.com/gexahedron/graph-puzzles-lean.

1.2 Statement of AI use

The proofs in this paper have been produced in a continuous dialog with AI, building upon the work done in [U26][6] and some other unreleased work.

[6] Nikolay Ulyanov. *Graph Puzzles III.1: A proof of Sabidussi's compatibility conjecture with 4 colours*. 2026. arXiv:2607.13225.

2 A parity lemma for cyclic sequences

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{F}_2^2$, put

$$\langle x, y \rangle = x_1 y_2 + x_2 y_1, \qquad q(x) = x_1 x_2.$$

The bilinear form is symmetric and alternating, and

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle . \tag{1}$$

For an even-sized list of vectors in $\mathbb{F}_2^2$, each of the four vectors occurs an even number of times exactly when the vector sum and the sum of the q-values both vanish. Indeed, if n_{rs} is the multiplicity of (r, s) modulo two, these two conditions give $n_{10} = n_{11}$, $n_{01} = n_{11}$ and $n_{11} = 0$; the parity of the list length then gives $n_{00} = 0$.

The bilinear balancing lemma in [U26], Lemma 5.1, has the following consequence for a set of three choices.

Lemma 2.1. *Let U be a finite set and let Σ have three elements. For distinct $u, v \in U$, suppose that $\beta_{uv} : \Sigma \times \Sigma \to \mathbb{F}_2$ satisfies*

$$\beta_{uv}(r,s) = \beta_{vu}(s,r), \qquad \sum_{s\in\Sigma} \beta_{uv}(r,s) = 0.$$

There are choices $t_u \in \Sigma$ such that

$$\sum_{v\neq u} \beta_{uv}(t_u, t_v) = 0 \qquad (u \in U).$$

Proof. Identify Σ with $\mathbb{F}_2^2 \setminus \{0\}$. A function on these three vectors whose values sum to zero extends uniquely to a linear map $\mathbb{F}_2^2 \to \mathbb{F}_2$: two of the vectors form a basis, and the third is their sum. For each β_{uv}, the row sums vanish by hypothesis, and the column sums vanish by transpose symmetry and the row-sum condition for β_{vu}. Thus β_{uv} extends uniquely to a bilinear map $\mathbb{F}_2^2 \times \mathbb{F}_2^2 \to \mathbb{F}_2$, with transpose symmetry preserved. The bilinear balancing lemma of [U26, Lemma 5.1] provides nonzero vectors satisfying the displayed equations. Restricting to Σ gives the assertion. □

For a cyclic sequence $w_0, \dots, w_{n-1}$ of elements of U, put $I_u = \{i : w_i = u\}$. Join consecutive positions by edges, with indices read modulo n. We write x_i for the colour of the edge from position i to position $i+1$; the two edge colours incident with position i are therefore x_{i-1} and x_i.

Lemma 2.2. *Let U be a finite set, let Σ be a set of three elements, and let $w_0, \dots, w_{n-1}$ be a cyclic sequence of elements of U. For each $u \in U$ and $t \in \Sigma$, let $\Delta_{u,t} : I_u \to \mathbb{F}_2^2 \setminus \{0\}$ satisfy*

$$\sum_{i\in I_u} \Delta_{u,t}(i) = 0, \qquad \sum_{t\in\Sigma} \Delta_{u,t}(i) = 0. \tag{2}$$

There are choices $t_u \in \Sigma$ and edge colours $x_0, \dots, x_{n-1} \in \mathbb{F}_2^2$ such that

$$x_{i-1} + x_i = \Delta_{w_i, t_{w_i}}(i)$$

and, for every u and $z \in \mathbb{F}_2^2$, the number of incidences of colour z in $(x_{i-1}, x_i)_{i\in I_u}$ is even.

Proof. Use the order $0 < 1 < \cdots < n-1$ to define, for $u \neq v$,

$$\beta_{uv}(t,s) = \sum_{\substack{i\in I_u,\ j\in I_v \\ j<i}} \left\langle \Delta_{u,t}(i), \Delta_{v,s}(j) \right\rangle.$$

Every pair in $I_u \times I_v$ occurs in one of the two orders. Hence

$$\beta_{uv}(t,s) + \beta_{vu}(s,t) = \left\langle \sum_{i\in I_u} \Delta_{u,t}(i), \sum_{j\in I_v} \Delta_{v,s}(j) \right\rangle = 0.$$

The second condition in (2) also gives $\sum_s \beta_{uv}(t,s) = 0$. Choose t_u by lemma 2.1, and write $y_i = \Delta_{w_i,t_{w_i}}(i)$. Set

$$x_{-1} = 0, \qquad x_i = \sum_{j=0}^{i} y_j.$$

The sum of all y_i is zero, so $x_{n-1} = x_{-1}$ and the edge colours are consistent around the cyclic sequence. Their successive differences are the required nonzero vectors.

Fix u. The vector sum of the $2|I_u|$ incident edge colours is $\sum_{i\in I_u} y_i = 0$. By (1), their quadratic sum is

$$\begin{aligned}\sum_{i\in I_u}\bigl(q(x_{i-1}) + q(x_i)\bigr) &= \sum_{i\in I_u} q(y_i) + \sum_{\substack{i\in I_u\\ j<i}} \bigl\langle y_i, y_j\bigr\rangle \\ &= q\left(\sum_{i\in I_u} y_i\right) + \sum_{v\neq u} \beta_{uv}(t_u, t_v) = 0.\end{aligned}$$

In the second equality, the terms with both indices in I_u combine with the individual q-values by repeated application of (1). The parity criterion at the start of this section proves the assertion. □

The *boundary* of an edge set is the set of its odd-degree vertices. An edge set with boundary T is called a *T-join*. We will use the following standard criterion.

Lemma 2.3 (T-joins)**.** *Let R be a graph and let T be a set of its vertices. There is an edge set $J \subseteq E(R)$ with boundary T if and only if every component of R contains an even number of vertices of T.*

Proof. The sum of degrees in a component is even, which proves necessity. For sufficiency, pair the vertices of T within each component, join each pair by a path, and take the symmetric difference of the path edge sets. Each internal vertex contributes even degree and the resulting boundary is T. □

3 A prescribed circuit as a member

A $\mathbb{F}_2^2$-flow assigns a vector to each edge so that the sum of the incident values at each vertex is zero; a loop has two incidences. A flow is *nowhere zero* when every edge value belongs to $\mathbb{F}_2^2 \setminus \{0\}$.

Theorem 1.1 (Prescribed circuit)**.** *Let G be a connected cubic graph and let C be a circuit of G. The following conditions are equivalent.*

(i) *G has a five-cycle double cover with C as a member.*

(ii) *$G - V(C)$ has a proper 3-edge-colouring.*

(iii) *$G - E(C)$ has a proper 3-edge-colouring.*

(iv) *G/C has a nowhere-zero $\mathbb{F}_2^2$-flow.*

Proof. We first prove the equivalence of (ii), (iii) and (iv). A proper 3-edge-colouring of $G - E(C)$ restricts to $G - V(C)$. Conversely, at each vertex outside C, assign its unused colours to the edges joining it to C. In $G - E(C)$ every vertex of C has degree one. Assign any colour to each chord of C. This extends a colouring of $G - V(C)$ to a proper colouring of $G - E(C)$.

Identify the three colours with the nonzero vectors of $\mathbb{F}_2^2$. A proper colouring of $G - E(C)$ satisfies flow conservation at every vertex outside C, since three distinct nonzero vectors sum to zero. Summing over those vertices also gives conservation at the contracted vertex of G/C. Conversely, a nowhere-zero $\mathbb{F}_2^2$-flow on G/C has three distinct values at every cubic vertex outside C, and therefore colours $G - E(C)$ properly. This proves the equivalence of conditions (ii)–(iv).

Assume (iii), and let f be the resulting nonzero edge values outside C. Write $C = v_0e_0v_1e_1 \dots v_{n-1}e_{n-1}v_0$, where $e_i = v_iv_{i+1}$, and let a_i be the unique edge outside C at v_i. For a component B of $G - E(C)$, let $I_B = \{i : v_i \in V(B)\}$. Summing conservation over the vertices of B outside C gives

$$\sum_{i \in I_B} f(a_i) = 0. \tag{3}$$

A chord is counted at both of its ends in this sum. Connectivity of G ensures that each component B has an attachment to C.

The linear map $\rho(r, s) = (s, r + s)$ has order three and satisfies $1 + \rho + \rho^2 = 0$. Apply lemma 2.2 to the sequence whose ith element is the component containing v_i, using $\Sigma = \{0, 1, 2\}$ and the assignments

$$\Delta_{B,t}(i) = \rho^t f(a_i).$$

Equation (3) and $1 + \rho + \rho^2 = 0$ give the two conditions in (2). Apply the chosen ρ^{t_B} to every edge value in B, and give e_i the colour x_i supplied by the lemma. The resulting assignment φ is a $\mathbb{F}_2^2$-flow on G: at v_i, conservation is $x_{i-1} + x_i + \rho^{t_B} f(a_i) = 0$.

Let $Z_0 = \{e : \varphi(e) = 0\}$. All these edges lie on C, and they form a matching because consecutive edge colours differ. By lemma 2.2, each component B of $G - E(C)$ contains an even number of endpoints

of Z_0. By lemma 2.3, choose $J \subseteq E(G) \setminus E(C)$ whose boundary is the endpoint set of Z_0. Then

$$Y = Z_0 \cup J, \qquad Z = C \triangle Y$$

are even edge sets, $Y \cap C = Z_0$, and $Z \subseteq E(G - Z_0)$.

This is the parity condition in Hušek and Šámal [HS26][7], Theorem 3.16, applied to the nowhere-zero $\mathbb{F}_2^3$-flow $(\mathbf{1}_C, \varphi)$, where $\mathbf{1}_C$ is the edge indicator of C. Its edges of value $(1,0,0)$ are precisely Z_0. We now apply Liu et al. [LHLZ23, Theorem 1.6] with $Q_0 = C$ and $Q_1 = Z$, since $E(C) \setminus E(Z) = Z_0$. This is the same construction as Hoffmann-Ostenhof's [HO13] with the two cycles C, Y; see also Xu [Xu14][8], Theorem 2.4. We spell out its five members below.

On $G - Z_0$, the flow φ is nowhere zero. The supports of φ_1, φ_2 and $\varphi_1 + \varphi_2$ are

$$D_1 = \operatorname{supp} \varphi_1, \quad D_2 = \operatorname{supp} \varphi_2, \quad D_3 = \operatorname{supp}(\varphi_1 + \varphi_2).$$

They are even, and every edge of $G - Z_0$ belongs to exactly two of them. Consequently

$$Z, \quad Z \triangle D_1, \quad Z \triangle D_2, \quad Z \triangle D_3 \tag{4}$$

double-cover $G - Z_0$: an edge of Z occurs in one of the three symmetric differences and in Z itself, while any other edge occurs in two of the symmetric differences. Replacing the first member by C and Y gives

$$C, \quad Y, \quad Z \triangle D_1, \quad Z \triangle D_2, \quad Z \triangle D_3. \tag{5}$$

On $G - Z_0$, membership in C and Y has the same total as membership in Z, since $C \cap Y = Z_0$. Every edge of Z_0 belongs to C and Y. Thus (5) is a five-cycle double cover with C as a member.

Finally, assume (i) and write the cover as C, A_1, A_2, A_3, A_4, adjoining empty members if needed. For $e \notin C$, put $d_i(e) = 1$ when $e \in A_i$ and $d_i(e) = 0$ otherwise, and assign

$$f(e) = \big(d_1(e) + d_2(e),\ d_1(e) + d_3(e)\big).$$

Exactly two of the four $d_i(e)$ equal one, so $f(e)$ is nonzero. Each coordinate is the indicator of a symmetric difference of even edge sets. Conservation therefore holds at every vertex outside C. At such a vertex the three nonzero values are distinct. Restriction to $G - V(C)$ gives the proper colouring in (ii). □

[7] Radek Hušek and Robert Šámal. *Exponentially many circuit double covers.* 2026. arXiv:2607.24724.

[8] Rui Xu. *Strong 5-cycle double covers of graphs.* Graphs and Combinatorics 30(2) (2014), 495–499. doi:10.1007/s00373-012-1266-8.

Theorem 1.2 (Critical and permutation graphs)**.** *Every connected critical cubic graph and every cubic permutation graph has a five-cycle double cover with any prescribed circuit as a member. In particular, critical snarks and permutation snarks have the strong five-cycle double cover property.*

Proof. Let G be a graph in either of the two classes, and let C be an arbitrary circuit of G. By theorem 1.1, it suffices to give a proper 3-edge-colouring of $G - V(C)$.

Critical graphs. Suppose that G is critical. Choose an edge uv of C. By criticality, $G - \{u, v\}$ has a proper 3-edge-colouring. Since $u, v \in V(C)$, the graph $G - V(C)$ is a subgraph of $G - \{u, v\}$, and the restriction of this colouring gives the required colouring.

Permutation graphs. Suppose that G is a cubic permutation graph, and write its spanning 2-factor as $A \cup B$, where A and B are induced circuits. Put $M = E(G) \setminus (E(A) \cup E(B))$. Cubicity makes M a perfect matching. Since both circuits are induced, every edge of M joins a vertex of A to a vertex of B.

If $C = A$ or $C = B$, deleting $V(C)$ also deletes every edge of M, and $G - V(C)$ is the other circuit. Every circuit has a proper 3-edge-colouring: use two alternating colours when its length is even; when its length is odd, alternate colours 1 and 2 on all but one edge and give the remaining edge colour 3.

If C is neither A nor B, it meets both of them. Indeed, an induced circuit contains no circuit other than itself, so C cannot lie wholly in $V(A)$ or wholly in $V(B)$. Deleting $V(C)$ therefore breaks both A and B into paths and isolated vertices. Colour the edges of each surviving path alternately with colours 1 and 2, and give every surviving edge of M colour 3. At each vertex, the path edges have different colours when both are present, and there is at most one matching edge. Thus this is a proper 3-edge-colouring of $G - V(C)$.

In each case, theorem 1.1 gives a five-cycle double cover with C as a member. In particular, this proves the assertion for every critical snark and every permutation snark, and hence their strong five-cycle double cover property. □

4 The hexagon and its edge-colouring

Let G and H satisfy the hypotheses of theorem 1.3, and denote the prescribed proper 3-edge-colouring of $G - E(H)$ by f. Write $e_i = v_i v_{i+1}$ and $s_i = v_i x_i$ for the hexagon edges and spokes, with all indices modulo six. Thus

$$\big(f(s_0), \dots, f(s_5)\big) = (1, 1, 2, 2, 3, 3). \tag{6}$$

The outer endpoints x_i may coincide. When counting them for a parity argument, count each spoke separately.

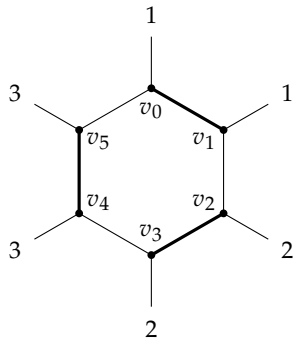


Figure 1: The spoke colours. The thick hexagon edges join equal-coloured spokes; the other three hexagon edges form U.

4.1 Three 2-factors

The colouring f gives three perfect matchings M_1, M_2, M_3. For $i \in \{1,2,3\}$, take the edges outside H of colour i, together with each of e_0, e_2, e_4 whose incident spokes have a colour different from i. Exactly one edge is selected at each hexagon vertex. Hence each M_i is a perfect matching. Put

$$B_i = E(G) \setminus M_i, \qquad U = \{e_1, e_3, e_5\}.$$

Each B_i is even. Edges outside H occur in two of the B_i; edges e_0, e_2, e_4 occur in one; and edges in U occur in all three.

Lemma 4.1. *If F is even and $F \cap E(H) = U$, then*

$$F, \quad H, \quad F \triangle B_1, \quad F \triangle B_2, \quad F \triangle B_3 \tag{7}$$

is a five-cycle double cover of G.

Proof. An edge outside H belongs to two of the B_i. If it belongs to F, it therefore belongs to exactly one of the three $F \triangle B_i$, and F supplies its second occurrence. If it does not belong to F, it belongs to two of the $F \triangle B_i$. Each edge of $H \setminus U$ belongs to one B_i and to H, and does not belong to F. Each edge of U belongs to F and H, and to none of the $F \triangle B_i$. All five members are even. □

In particular, an even set $F = C \cup D$ with D vertex-disjoint from C makes C a component of the first member of (7). We next construct D in the intersection case that requires it.

4.2 A circuit meeting the hexagon

Let C be a circuit meeting H. At each common vertex, the two pairs of edges used by C and H share an edge, by cubicity. Thus every

common vertex has a neighbour that is also common. There are either at least three common vertices or exactly the ends of one shared edge.

Suppose first that $|V(C) \cap V(H)| \geq 3$. The surviving part of H in $G - V(C)$ consists of paths on at most three vertices. Retain f outside H. A single edge can avoid both of its endpoint spoke colours. On a three-vertex path, let its spoke colours be a, b, c. For $a = b$, choose the edge at the opposite end first, avoiding b, c, and give the other edge the remaining colour different from a, b. The case $b = c$ is symmetric. For three distinct colours, give the first edge colour c and the second colour a. Every consecutive triple in (6) has one of these forms. A deleted spoke allows the same colouring to be restricted to the surviving edges. Thus $G - V(C)$ is 3-edge-colourable, and theorem 1.1 gives a five-cycle double cover with C as a member.

If the two common vertices are v_j, v_{j+1} with j even, deleting them leaves a four-vertex path with spoke colours $2, 2, 3, 3$, after cyclic relabelling of the colours. Its three edges can be coloured $3, 1, 2$. Again theorem 1.1 applies.

It remains to treat two common vertices at an edge of U. Rotate the indices and permute the colours so that the shared edge is e_0 and the cyclic list of spoke colours becomes

$$(1, 2, 2, 3, 3, 1). \tag{8}$$

In these coordinates, $U = \{e_0, e_2, e_4\}$ and the four surviving spokes have outer endpoints x_2, x_3, x_4, x_5. Let $R = G - V(C \cup H)$.

Lemma 4.2. *In the coordinates of* (8)*, one of the following exists.*

(i) *A proper 3-edge-colouring of* $G - V(C)$.

(ii) *An even subgraph* D*, vertex-disjoint from* C*, such that* $D \cap E(H) = \{e_2, e_4\}$.

Proof. First suppose that all four x_2, x_3, x_4, x_5 lie outside C, and that every component of R contains an even number of these four endpoints, counted with repetition. The edge set

$$Q = \{s_2, s_3, s_4, s_5, e_2, e_4\}$$

has even degree at each hexagon vertex, and its boundary in R is the set of vertices occurring an odd number of times among x_2, x_3, x_4, x_5. By lemma 2.3, an edge set $J \subseteq E(R)$ has this same boundary. The union $D = Q \cup J$ is even, avoids C, and has the asserted hexagon edges.

The other configurations admit a recolouring of the four-vertex path. The following table changes just one of its spoke colours. The last column gives the colours of e_2, e_3, e_4 in that order.

Endpoint	New spoke colours at v_2, v_3, v_4, v_5	Path colours
x_2	$1, 3, 3, 1$	$2, 1, 2$
x_3	$2, 2, 3, 1$	$3, 1, 2$
x_4	$2, 3, 1, 1$	$1, 2, 3$
x_5	$2, 3, 3, 2$	$1, 2, 1$

Each row gives three distinct colours at the two internal path vertices and a path-edge colour different from the endpoint spoke colour. If some x_t lies on C, its spoke is deleted from $G - V(C)$. Use the row for x_t and retain f on every surviving edge outside H.

Otherwise all four endpoints lie in R, and some component contains an odd number of them, counted with repetition. That component, or the union of the other components, contains exactly one of the four, say the one at x_t. Apply throughout those components the transposition of the original colour of s_t and the new colour in the corresponding table row. Apply the same transposition to all edges from these components to H. Properness is preserved at every vertex of R, and among the four surviving spokes exactly s_t changes colour. The table now completes a proper colouring of $G - V(C)$. □

In the first case of lemma 4.2, theorem 1.1 gives a five-cycle double cover with C as a member. In the second, C uses exactly e_0 of the hexagon, so $F = C \cup D$ satisfies the hypothesis of lemma 4.1. The resulting cover has C as a component. This settles every circuit meeting H.

5 A circuit disjoint from the hexagon

Keep G, H and f as in section 4. Let C be a circuit of G vertex-disjoint from H, and put $R = G - V(C \cup H)$. We use two elementary operations on covers before considering the possible connections among the six spokes.

Lemma 5.1. *Let K be a properly 3-edge-coloured cubic graph, and let B_i be the union of its two colour classes other than i. If C, D are vertex-disjoint even subgraphs, then*

$$C, \quad D, \quad B_1 \triangle C \triangle D, \quad B_2 \triangle C \triangle D, \quad B_3 \triangle C \triangle D \tag{9}$$

is a five-cycle double cover of K.

Proof. Every edge belongs to two of the B_i. An edge of $C \cup D$ therefore belongs to exactly one of the three $B_i \triangle C \triangle D$; C or D supplies its second occurrence. Every other edge belongs to two of these symmetric differences. All five members are even. □

Label an edge of a double cover by the pair of indices of the members containing it. Evenness of each member means that every index occurs an even number of times at each vertex.

Lemma 5.2 (Restoring the hexagon)**.** *Suppose that the edges outside H are labelled by pairs of cover indices, with each index occurring an even number of times at every vertex outside H. Suppose that the six spoke labels, in order, are*

$$\{s,a\},\{s,a\},\{s,b\},\{s,b\},\{s,c\},\{s,c\},$$

where s,a,b,c are distinct. Introduce a fresh index h, and label the hexagon edges $e_0,\dots,e_5$ by

$$\{a,h\},\{s,h\},\{b,h\},\{s,h\},\{c,h\},\{s,h\}. \tag{10}$$

These labels define a double cover of G, with H itself as member h. If C is already a member, is disjoint from H, and has an index different from s,a,b,c, then C remains a member.

Proof. At each hexagon vertex the three incident pairs are the three sides of a triangle on three indices. Every index therefore has degree zero or two there. All edges receive exactly two indices, and h occurs precisely on H. □

In the applications below, the initial cover has five members and contains C as a member. After lemma 5.2 adds H as a member, replace C and H by $C \cup H$. This gives a five-cycle double cover with C as a component of one member.

5.1 Connecting an equal-coloured pair

Suppose that some pair of spokes with the same colour has its outer endpoints in the same component of R. By an even rotation and a colour permutation, take this pair to be s_0, s_1, of colour 1. Delete H, and replace the three equal-coloured spoke pairs by new edges x_0x_1, x_2x_3, x_4x_5, coloured 1, 2, 3, respectively. The resulting cubic multigraph K is properly coloured by the extended f. Each new edge has distinct ends, since equal-coloured spokes have distinct outer endpoints in a proper colouring.

Choose a path in R from x_0 to x_1. Together with the first edge it forms a circuit D of K disjoint from C. Apply lemma 5.1, with member indices $\mathsf{c},\mathsf{d},1,2,3$ for C, D and the three symmetric differences. The first new edge has label $\{\mathsf{d},1\}$. The second and third new edges have labels $\{1,3\}$ and $\{1,2\}$. Replace each new edge by its two original spokes, giving both spokes that edge's label. Their ordered labels are

$$\{1,\mathsf{d}\},\{1,\mathsf{d}\},\{1,3\},\{1,3\},\{1,2\},\{1,2\}.$$

Apply lemma 5.2 with $s = 1$, $a = \mathsf{d}$, $b = 3$, $c = 2$. Replace the members C and H by their union to obtain the required five-cycle double cover.

5.2 Permuting colours in the remaining components

We may now assume that the two outer endpoints of each equal-coloured pair belong to different components of R, or that at least one of them lies on C. Partition the positions $0, \dots, 5$ according to the components of R containing their outer endpoints. Give each spoke directly joining H to C its own singleton block. No block contains $\{0,1\}$, $\{2,3\}$ or $\{4,5\}$.

On each component of R, we may independently permute the three colours, including on its incident spokes to H. The proper colouring is preserved on R. A spoke to C is deleted from $G - V(C)$, so its colour in the local completion may be chosen freely.

Lemma 5.3. *Let $\mathcal{P}$ be a partition of $\{0, \dots, 5\}$ with each of $\{0,1\}$, $\{2,3\}$ and $\{4,5\}$ split between blocks. Starting from the spoke colours $1,1,2,2,3,3$, independent colour permutations on the blocks give spoke colours that extend to a proper colouring of the hexagon whenever*

$$\mathcal{P} \neq \big\{\{0,3\},\{1,4\},\{2,5\}\big\}.$$

Proof. Coarsen $\mathcal{P}$ until every pair of distinct blocks is joined by one of the three original pairs $\{0,1\}$, $\{2,3\}$ and $\{4,5\}$. Such a coarsening preserves the hypothesis. Regard the original pairs as three edges on the set of blocks. The resulting underlying graph is complete, so it has two blocks or three blocks forming a triangle. The twelve possibilities are listed below. In the first column, positions carrying the same digit belong to the same block; digits are assigned in order of first occurrence.

Partition	Spoke colours	Colours of $e_0, \dots, e_5$
010101	112332	231213
010110	122133	313212
010212	112332	231213
010221	122133	313212
011001	132231	213123
011010	113223	321312
011202	112332	231213
011220	123123	312312
012012	opposite-pair partition	
012021	113223	321312
012102	122133	313212
012120	113223	321312

For each displayed colouring, equalities and inequalities of the original spoke colours are preserved within every block. Thus each block

assignment extends to a permutation of the three colours. At each hexagon vertex the spoke and its two incident hexagon edges have the three distinct colours. The same assignment works for every refinement of that block partition.

If $\mathcal{P}$ is a proper refinement of the opposite-pair partition, some opposite pair is split. An even rotation places that pair at $\{0,3\}$ and permutes the original three spoke colours. The partition then also refines $\{\{0,2,5\},\{1,3,4\}\}$, represented by 010110. Its row gives the required completion. □

Whenever lemma 5.3 applies, combine its local completion with the permuted colourings of R, and restrict to $G - V(C)$. This gives a proper 3-edge-colouring. By theorem 1.1, the circuit C is a member of a five-cycle double cover.

5.3 Three pairs of opposite vertices

The remaining partition has three blocks $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$. All six spokes end in R, and each opposite pair has its endpoints in its own component. Replace H by a claw with centre z and leaves w_0, w_1, w_2. Attach the spokes formerly at v_i, v_{i+3} to w_i, for $i = 0,1,2$. Their colour pairs are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. Colour the central edges zw_0, zw_1, zw_2 with $3,2,1$, respectively. This gives a properly coloured cubic multigraph K.

In each of the three components of R, choose a path between the two endpoints of its opposite spokes. The path and the two spoke edges at w_i form a circuit D_i. Coincident endpoints give a path of length zero and a two-edge circuit. The three circuits are vertex-disjoint and avoid C. Set $D = D_0 \cup D_1 \cup D_2$ and apply lemma 5.1, again with indices $\mathsf{c}, \mathsf{d}, 1, 2, 3$.

Every spoke edge belongs to D. A spoke of original colour j therefore has cover label $\{\mathsf{d}, j\}$. Delete the claw and restore the original six spokes. Their labels, in hexagon order, are

$$\{\mathsf{d},1\},\{\mathsf{d},1\},\{\mathsf{d},2\},\{\mathsf{d},2\},\{\mathsf{d},3\},\{\mathsf{d},3\}.$$

Use lemma 5.2 with $s = \mathsf{d}$, $a = 1$, $b = 2$, $c = 3$. Replace the members C and H by their union to obtain a five-cycle double cover with C as a component. Every circuit disjoint from H has now been treated.

6 Strong covers

Theorem 1.3 (An induced hexagon)**.** *Let G be a connected cubic graph with an induced hexagon $H = v_0v_1 \dots v_5v_0$. Suppose that $G - E(H)$ has a proper 3-edge-colouring in which the spokes at $v_0, \dots, v_5$ have colours $1, 1, 2, 2, 3, 3$, respectively. Then every circuit of G is a component of a member of a five-cycle double cover.*

Proof. Let G and H satisfy the hypotheses, and let C be an arbitrary circuit of G. If C meets H, the constructions in section 4 give either a five-cycle double cover with C as a member or the cover of lemma 4.1 with $C \cup D$ as one member, where D is vertex-disjoint from C. If C is vertex-disjoint from H, section 5 gives a five-cycle double cover with C as a member or a cover having $C \cup H$ as one member. Thus every circuit is a component of a member of a five-cycle double cover. □

Corollary 1.4. *Every snark of colouring defect three has the strong five-cycle double cover property.*

Proof. Let G be a snark with $\mathrm{df}(G) = 3$. By [KMNS24], Theorem 3.3, G has an induced hexagon H with a proper 3-edge-colouring of $G - E(H)$ whose spoke colours are $1, 1, 2, 2, 3, 3$, after a rotation and colour permutation. Apply theorem 1.3. □

For an individual circuit C in a snark of colouring defect three, theorem 1.1 also decides whether C itself can be a member of a five-cycle double cover: this holds if and only if $G - V(C)$ is 3-edge-colourable.

A Relationships between snark classes

For context, we recall how the classes considered here relate to two further familiar classes. A snark G is *hypohamiltonian* if $G - v$ has a Hamiltonian circuit for every vertex v (while G itself has none). It is *bicritical* if $G - \{u, v\}$ is 3-edge-colourable for every pair of distinct vertices u, v, rather than only for adjacent pairs as in the definition of criticality.

Every hypohamiltonian snark is bicritical, and every bicritical snark is critical. Both inclusions are strict. Steffen [So01][9], Theorem 2.4 and the following remark, records the first inclusion and the bicritical, non-hypohamiltonian Goldberg snarks on 22 vertices. Critical snarks that are not bicritical are discussed by Máčajová and Škoviera [MS21][10], Section 3; such examples occur at every even order at least 32. Bicritical snarks are also precisely the irreducible snarks [MS21, Theorem 6].

Permutation snarks are conjecturally bicritical. Equivalently, the conjecture asks whether every permutation snark is critical. Máčajová and Škoviera [MS25][11] study the criticality question and report that all permutation snarks with fewer than 50 vertices are bicritical. Their Theorem 1.1 shows that a noncritical permutation snark would imply the existence of a permutation snark of order 6 (mod 8).

The equivalence concerns the two assertions about the whole class. Indeed, in a permutation snark a pair whose deletion leaves an uncolourable graph must lie on the same defining circuit: deleting one vertex from each circuit leaves two paths, which can be coloured alternately with two colours, with the matching coloured by the third. If the pair on one circuit is nonadjacent, reconnect that circuit to make the pair adjacent, changing only one other edge outside the pair. The resulting graph is still a permutation graph, and deleting the pair leaves a graph containing the original uncolourable remainder. Thus a counterexample to the bicritical formulation yields a counterexample to the critical formulation.

We also consider the class of *non-strong* snarks. A snark is *strong* if deleting any edge and suppressing the resulting vertices of degree two always leaves an uncolourable cubic graph. Equivalently, $G - \{u, v\}$ is uncolourable for every edge uv [MS21, Theorem 7]. Thus a snark is non-strong precisely when $G - \{u, v\}$ is 3-edge-colourable for at least one edge uv. Every critical snark is therefore non-strong. The two-path colouring above shows that every permutation snark is non-strong as well.

[9] Eckhard Steffen. *On bicritical snarks.* Mathematica Slovaca 51(2) (2001), 141–150. DML-CZ:136800.

[10] Edita Máčajová and Martin Škoviera. *Critical and flow-critical snarks coincide.* Discussiones Mathematicae Graph Theory 41(2) (2021), 503–511. doi: 10.7151/dmgt.2204.

[11] Edita Máčajová and Martin Škoviera. *Are there any permutation snarks on* 6 (mod 8) *vertices?* Procedia Computer Science 273 (2025), 163–168. doi:10.1016/j.procs.2025.10.294.

Karabáš, Máčajová, Nedela and Škoviera [KMNS25][12], Theorem 1.1, prove that every strong snark has colouring defect at least six. In particular, every snark of colouring defect at most four is non-strong.

The earlier defect-three conjectures are false. Karabáš, Máčajová, Nedela and Škoviera [KMNS24, Conjectures 7.4 and 7.6] conjectured that every critical snark, and every bicritical snark, has colouring defect three. Steffen [S15][13], Conjecture 4.1, made the assertion for hypohamiltonian snarks (his μ_3 is our df). The author's construction [U26D][14] gives, for each integer $t \geq 0$, a hypohamiltonian permutation snark G_t with

$$|V(G_t)| = 58 + 24t, \qquad \mathrm{df}(G_t) = 4.$$

Each G_t has girth five, cyclic edge-connectivity four, and no hexagon. The 58-vertex member is available as House of Graphs 58070. Since every G_t is also bicritical and critical, the family refutes all three conjectures and also shows that permutation snarks need not have defect three. We propose the following weakened bound.

Conjecture A.1. *Every critical snark and every permutation snark has colouring defect at most four. In particular, every bicritical snark and every hypohamiltonian snark has colouring defect at most four.*

The family G_t shows that four would be best possible in each of these four classes. The permutation assertion is stated independently of the permutation criticality conjecture.

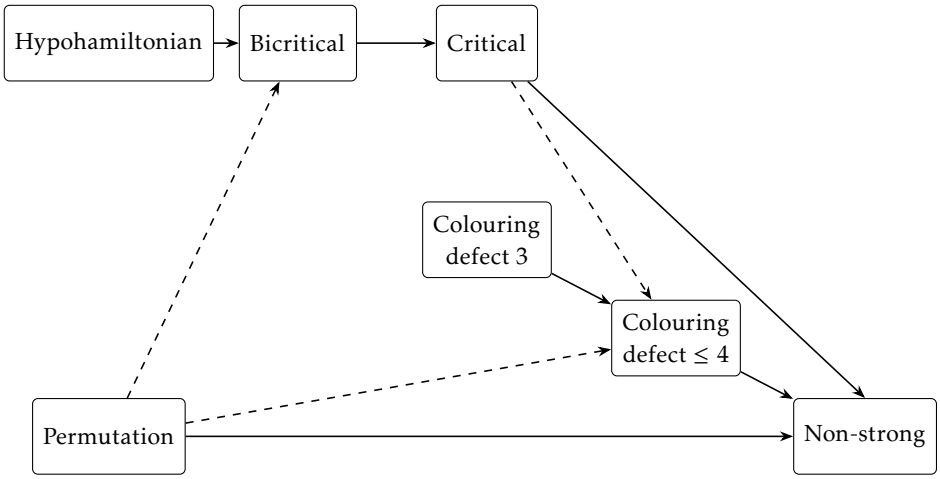


Figure 2: Arrows point to containing classes. Solid arrows are proved inclusions; dashed arrows are conjectural. The two arrows to defect at most 4 express conjecture A.1. The permutation-to-bicritical arrow can equivalently be directed to critical.

Defect three does not imply criticality, even for cyclically 4-edge-connected snarks of girth at least five: the six snarks on 20 vertices in [KMNS24, Table 1] all have defect three, but only one is critical.

[12] Ján Karabáš, Edita Máčajová, Roman Nedela and Martin Škoviera. *Colouring defect of strong snarks*. Procedia Computer Science 273 (2025), 149–155. doi:10.1016/j.procs.2025.10.292.

[13] Eckhard Steffen. *1-factor and cycle covers of cubic graphs*. Journal of Graph Theory 78(3) (2015), 195–206. doi:10.1002/jgt.21798.

[14] Nikolay Ulyanov. *Hypohamiltonian permutation snarks of colouring defect four*. 2026, manuscript. The 58-vertex example is House of Graphs 58070.

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