

Graph Puzzles III.1:

A proof of Sabidussi's compatibility conjecture

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ABSTRACT: We prove Sabidussi's compatibility conjecture. Let G be a finite connected multigraph in which every vertex has even degree and the minimum degree is at least four, and let T be a closed trail that traverses every edge exactly once. The edges of G can be partitioned into circuits (connected 2-regular subgraphs) so that no circuit contains the two edges used consecutively anywhere in T . In fact, the edges can be four-coloured so that every such pair receives two different colours and the subgraph formed by the edges of each colour has even degree at every vertex.

1 Introduction

All graphs in this paper are finite multigraphs; loops and parallel edges are allowed. We regard an edge as having two distinct *half-edges*, each incident with a vertex, so that a loop contributes two to the degree of its vertex. A graph is *even* if every vertex has even degree and *Eulerian* if it is connected and even. A *circuit* is a nonempty connected 2-regular subgraph, and a *circuit decomposition* is a partition of the edge set into circuits.

A transition at a vertex is an unordered pair of distinct half-edges incident with that vertex. A transition system pairs the half-edges at each vertex. Every circuit pairs its two half-edges at each vertex that it meets; it is *compatible* with a transition system if none of these pairs is a prescribed transition. A circuit decomposition is compatible if each of its circuits is compatible. An Euler tour induces a transition system by pairing, at each passage through a vertex, the half-edge on which the tour arrives with the half-edge on which it departs.

We prove the following strengthening of Sabidussi's conjecture.

Theorem 1.1. *Let G be a finite Eulerian multigraph with minimum degree $\delta(G) \geq 4$, and let $T = (e_0, e_1, \dots, e_{m-1})$ be an Euler tour of G , with indices read modulo m . There is a colouring $\chi : E(G) \rightarrow \mathbb{F}_2^2$ such that*

1. $\chi(e_{i-1}) \neq \chi(e_i)$ for every i ;
2. every vertex has even degree in each colour class.

Consequently, G has a circuit decomposition compatible with the transition system induced by T .

For the following consequence, we use the standard bounded-cover terminology: a *cycle* is an even subgraph and need not be connected, and a *k -cycle double cover* is a family of at most k cycles in which every edge occurs exactly twice.

The cycle double cover conjecture was recently proved by OpenAI [Ope26]. Under the convention above, their construction proves the 8-cycle double cover theorem: the eight even subgraphs indexed by $\mathbb{F}_2^3$ cover every edge exactly twice. In the presence

of a prescribed dominating circuit, theorem 1.1 improves the bound and retains that circuit as a member.

Corollary 1.2. *Let H be a finite cubic graph and let C be a dominating circuit of H . Then H has a 5-cycle double cover with C as a member.*

Proof. Contract every component of $H - E(C)$. The image of C is an Euler tour T of an Eulerian multigraph G whose degrees are either 4 or 6. The colour classes given by theorem 1.1 partition $E(G)$ into at most four compatible even subgraphs. Under the standard lift through the contraction [Fle84, pp. 236–237], these become at most four cycles of H that cover the edges of C once and the edges outside C twice. Adjoining C gives the required cover. $\square$

The conjecture, due to Gert Sabidussi, was recorded by Fleischner [Fle80]. The study of forbidden transitions goes back to Kotzig [Kot68]. A compatible circuit decomposition need not exist for an arbitrary transition system: the standard example is K_5 with the transition system induced by a decomposition into two Hamiltonian circuits [FBZZ19]. Fleischner proved the planar case of Sabidussi’s conjecture [Fle80], and Fleischner and Frank later established a more general planar decomposition theorem [FF90]. Fan and Zhang proved the K_5 -minor-free case [FZ00]; subsequent work developed the corresponding theory of transition minors [FBZZ19]. Fleischner also showed that it is enough to consider Eulerian graphs all of whose degrees are either 4 or 6 [Fle84, Lemma 1, p. 236]. Our proof instead treats every even degree directly.

The argument works with the stronger colouring in theorem 1.1. The successive passage vertices of the Euler tour form a cyclic word whose gaps are the tour edges. After passing from gap colours x_i to their nonzero differences $y_i = x_{i-1} + x_i$, each letter has three natural local zero-sum patterns. A parity lemma chooses one pattern at every letter so that all remaining quadratic conditions vanish simultaneously. Prefix sums then recover the gap colours. The two elementary parity facts used in this construction are isolated in section 3.

The proof has also been formalized in Lean. The formalization is available in the author’s GitHub repository: <https://github.com/gexahedron/sabidussi-lean>.

Statement of AI use. The proof in this paper is entirely due to GPT 5.6 Pro, and the writeup was prepared with help from GPT 5.6 Sol.

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2 Reduction to a cyclic word

Fix G and T as in theorem 1.1. Temporarily orient every edge e_i in the direction in which the tour traverses it, and denote its departure and arrival half-edges by e_i^- and e_i^+ , respectively. Let v_i be the vertex at which the tour passes from e_{i-1} to e_i . Thus the transition at this passage is $\{e_{i-1}^+, e_i^-\}$. Since $\delta(G) \geq 4$, the tour has at least two edges, so its consecutive edges e_{i-1} and e_i are distinct.

For $v \in V(G)$, put $I_T(v) = \{i \in \{0, 1, \dots, m-1\} : v_i = v\}$. If $H_G(v)$ denotes the

set of half-edges incident with v , then

$$\begin{aligned} I_T(v) \times \{\text{in}, \text{out}\} &\longrightarrow H_G(v), \\ (i, \text{in}) &\longmapsto e_{i-1}^+, \\ (i, \text{out}) &\longmapsto e_i^- \end{aligned}$$

is a bijection. Indeed, a half-edge at v occurs exactly once in the tour, either as some departure e_j^- , in which case $v_j = v$, or as some arrival e_j^+ , in which case $v_{j+1} = v$. The two half-edges of a loop remain distinct in this correspondence. Consequently,

$$|I_T(v)| = \frac{\deg_G(v)}{2} \geq 2. \quad (1)$$

Definition (Cyclic words). A *cyclic word* of positive length n on an alphabet A is a sequence

$$\mathbf{w} = (w_0, w_1, \dots, w_{n-1}),$$

read cyclically, in which every letter of A occurs. Put

$$I_a = \{i \in \{0, 1, \dots, n-1\} : w_i = a\} \quad (a \in A).$$

Gap i lies immediately after w_i ; hence occurrence i is incident with gaps $i-1$ and i . When the same gap is incident twice, the two incidences are counted separately.

The combinatorial core of the proof is the following theorem.

Theorem 2.1 (Cyclic-word colouring). *Let $\mathbf{w}$ be a cyclic word in which every letter occurs at least twice. Its gaps can be coloured by elements $x_0, x_1, \dots, x_{n-1} \in \mathbb{F}_2^2$ so that*

1. $x_{i-1} \neq x_i$ for every i ;
2. for every letter a and every $z \in \mathbb{F}_2^2$, the colour z occurs an even number of times among the gap incidences $(x_{i-1}, x_i)_{i \in I_a}$.

We next derive the graph theorem from this statement. Since $\delta(G) > 0$ and G is connected, every vertex occurs among the passage vertices. Thus $\mathbf{w}_T = (v_0, v_1, \dots, v_{m-1})$ is a cyclic word on $V(G)$, and it satisfies the hypothesis of theorem 2.1 by eq. (1). Given the gap colouring supplied by that theorem, define $\chi(e_i) = x_i$. The two edges in the transition at passage i receive the distinct colours x_{i-1} and x_i . Moreover, the displayed bijection identifies the half-edges of colour z at v with the gap incidences of colour z at occurrences of v . The second conclusion of theorem 2.1 therefore says exactly that every colour class is even.

It remains only to pass from colour classes to circuits. Every finite even multigraph has a circuit decomposition: take an Euler tour in each component containing an edge and split it at repeated vertices until every resulting closed trail is a circuit. Apply this observation to the four colour classes. The resulting circuits partition $E(G)$, and each is monochromatic, whereas the two edges in every transition have different colours. The decomposition is therefore compatible with the transition system induced by T . Thus theorem 2.1 implies theorem 1.1.

3 Two parity lemmas

All calculations in this section take place in $\mathbb{F}_2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{F}_2^2$, put

$$\langle x, y \rangle = x_1 y_2 + x_2 y_1, \quad q(x) = x_1 x_2. \quad (2)$$

The first form is bilinear, symmetric, and alternating, and

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle. \quad (3)$$

Iterating this identity gives, for every finite sequence $z_1, \dots, z_r \in \mathbb{F}_2^2$,

$$q\left(\sum_{i=1}^r z_i\right) = \sum_{i=1}^r q(z_i) + \sum_{1 \leq j < i \leq r} \langle z_i, z_j \rangle. \quad (4)$$

Lemma 3.1 (Four-colour parity). *Let $(z_i)_{i \in I}$ be a finite family in $\mathbb{F}_2^2$, with $|I|$ even. Every element of $\mathbb{F}_2^2$ occurs an even number of times among the z_i if and only if*

$$\sum_{i \in I} z_i = 0 \quad \text{and} \quad \sum_{i \in I} q(z_i) = 0. \quad (5)$$

Proof. If all four multiplicities are even, both sums in eq. (5) vanish. Conversely, let $n_{rs} \in \mathbb{F}_2$ be the parity of the multiplicity of $(r, s) \in \mathbb{F}_2^2$. Since $|I|$ is even,

$$n_{00} + n_{10} + n_{01} + n_{11} = 0.$$

The two coordinates of the first equation in eq. (5) give

$$n_{10} + n_{11} = 0, \quad n_{01} + n_{11} = 0.$$

Finally, q takes the value one only at $(1, 1)$, so the second equation gives $n_{11} = 0$. Hence all four multiplicities are even. $\square$

The second lemma is the only global selection principle needed in the proof.

Lemma 3.2 (Three-state balancing). *Let U be a finite set and let Σ be a three-element set. For each ordered pair of distinct elements $u, v \in U$, let*

$$\beta_{uv} : \Sigma \times \Sigma \longrightarrow \mathbb{F}_2$$

satisfy

$$\beta_{uv}(r, s) = \beta_{vu}(s, r), \quad (r, s \in \Sigma), \quad (6)$$

$$\sum_{s \in \Sigma} \beta_{uv}(r, s) = 0 \quad (r \in \Sigma). \quad (7)$$

For an assignment $\sigma : U \rightarrow \Sigma$, write $\sigma_u = \sigma(u)$. Then the number of assignments satisfying

$$\sum_{\substack{v \in U \\ v \neq u}} \beta_{uv}(\sigma_u, \sigma_v) = 0 \quad (u \in U) \quad (8)$$

is odd. In particular, such an assignment exists.

Proof. All sums below are in $\mathbb{F}_2$. For $\sigma \in \Sigma^U$, define

$$Q_u(\sigma) = \sum_{\substack{v \in U \\ v \neq u}} \beta_{uv}(\sigma_u, \sigma_v).$$

If N is the number of assignments satisfying eq. (8), write $\bar{N} \in \mathbb{F}_2$ for its parity. Then

$$\begin{aligned} \bar{N} &= \sum_{\sigma \in \Sigma^U} \prod_{u \in U} (1 + Q_u(\sigma)) \\ &= \sum_{S \subseteq U} Z(S), \quad Z(S) = \sum_{\sigma \in \Sigma^U} \prod_{u \in S} Q_u(\sigma). \end{aligned} \quad (9)$$

Here $1 + Q_u(\sigma)$ is the indicator that $Q_u(\sigma) = 0$. The term for the empty set is

$$Z(\emptyset) = 3^{|U|} = 1.$$

We prove that $Z(S) = 0$ for every nonempty S .

Expanding one summand from each Q_u , $u \in S$, gives

$$Z(S) = \sum_{\substack{f: S \rightarrow U \\ f(u) \neq u \ (u \in S)}} Z_f, \quad Z_f = \sum_{\sigma \in \Sigma^U} \prod_{u \in S} \beta_{uf(u)}(\sigma_u, \sigma_{f(u)}). \quad (10)$$

Draw an arc $u \rightarrow f(u)$ for every $u \in S$, and let $d_f(w)$ be the degree of w in the underlying undirected multigraph, with multiplicities retained. If $d_f(w) = 1$, the state σ_w occurs in exactly one factor of the product defining Z_f . Summing first over this state gives zero by eq. (7); when w is the source of its unique arc, use eq. (6) first. Hence $Z_f = 0$ whenever the functional digraph has a vertex of degree one.

Suppose now that it has no such vertex, and let $U_f = S \cup f(S)$ be the set of vertices incident with an arc. Every vertex of U_f has degree at least two. Since there are $|S|$ arcs and $S \subseteq U_f$,

$$2|S| = \sum_{w \in U_f} d_f(w) \geq 2|U_f| \geq 2|S|.$$

Equality holds throughout. Thus $U_f = S$ and every vertex of S has degree two. Each already has out-degree one, so it also has in-degree one; hence f is a fixed-point-free permutation of S .

The variables outside S contribute $3^{|U \setminus S|} = 1$, and the remaining sum factors over the cycles of f . A two-cycle $u \leftrightarrow v$ contributes

$$\begin{aligned} \sum_{r, s \in \Sigma} \beta_{uv}(r, s) \beta_{vu}(s, r) &= \sum_{r, s \in \Sigma} \beta_{uv}(r, s)^2 \\ &= \sum_{r, s \in \Sigma} \beta_{uv}(r, s) = 0, \end{aligned}$$

by eqs. (6) and (7). Thus $Z_f = 0$ if f has a two-cycle. Every remaining permutation has all cycles of length at least three and is distinct from its inverse. Reversing the cycles and using eq. (6) gives $Z_f = Z_{f^{-1}}$, so these terms cancel in inverse pairs. Therefore $Z(S) = 0$ for every nonempty S . Equation (9) now gives $\bar{N} = 1$, as required. $\square$

4 Proof of the cyclic-word theorem

Proof of theorem 2.1. Choose a representative $\mathbf{w} = (w_0, \dots, w_{n-1})$ and use the linear order $0 < 1 < \dots < n-1$. Put

$$\Omega = \mathbb{F}_2^2 \setminus \{0\}$$

and define

$$\rho : \mathbb{F}_2^2 \longrightarrow \mathbb{F}_2^2, \quad \rho(r, s) = (s, r + s).$$

The map ρ has order three and cyclically permutes the three elements of Ω . In particular,

$$\sum_{t \in \Omega} t = 0. \quad (11)$$

Local patterns. Fix a letter a , and write $I_a = \{i_1 < i_2 < \dots < i_k\}$, where $k \geq 2$. For each $t \in \Omega$, define $\Delta_{a,t} : I_a \rightarrow \Omega$ as follows. If k is even, put $\Delta_{a,t}(i_j) = t$ for every j . If k is odd, put

$$\Delta_{a,t}(i_j) = \begin{cases} t, & 1 \leq j \leq k-2, \\ \rho t, & j = k-1, \\ \rho^2 t, & j = k. \end{cases}$$

These patterns satisfy

$$\Delta_{a,t}(i) \neq 0, \quad (i \in I_a, t \in \Omega), \quad (12)$$

$$\sum_{i \in I_a} \Delta_{a,t}(i) = 0, \quad (t \in \Omega), \quad (13)$$

$$\sum_{t \in \Omega} \Delta_{a,t}(i) = 0, \quad (i \in I_a). \quad (14)$$

The first identity holds by construction. For the second, an even number of copies of t cancels when k is even. When k is odd, the first $k-3$ copies cancel in pairs and the final three values $t, \rho t, \rho^2 t$ sum to zero. For fixed i , the map $t \mapsto \Delta_{a,t}(i)$ is one of $1, \rho, \rho^2$ on Ω , so eq. (14) follows from eq. (11).

Interactions. For distinct letters $a, b \in A$, define

$$\beta_{ab}(t, s) = \sum_{\substack{i \in I_a, j \in I_b \\ j < i}} \langle \Delta_{a,t}(i), \Delta_{b,s}(j) \rangle \quad (t, s \in \Omega).$$

These functions satisfy the hypotheses of lemma 3.2. Indeed, every pair $(i, j) \in I_a \times I_b$ occurs in exactly one of the two orders $j < i$ and $i < j$. Hence symmetry and bilinearity give

$$\begin{aligned} \beta_{ab}(t, s) + \beta_{ba}(s, t) &= \sum_{i \in I_a} \sum_{j \in I_b} \langle \Delta_{a,t}(i), \Delta_{b,s}(j) \rangle \\ &= \left\langle \sum_{i \in I_a} \Delta_{a,t}(i), \sum_{j \in I_b} \Delta_{b,s}(j) \right\rangle = 0 \end{aligned}$$

by eq. (13). For fixed t ,

$$\sum_{s \in \Omega} \beta_{ab}(t, s) = \sum_{\substack{i \in I_a, j \in I_b \\ j < i}} \left\langle \Delta_{a,t}(i), \sum_{s \in \Omega} \Delta_{b,s}(j) \right\rangle = 0$$

by eq. (14).

Apply lemma 3.2 with $U = A$ and $\Sigma = \Omega$. It supplies a state $t_a \in \Omega$ for every letter a such that

$$\sum_{\substack{b \in A \\ b \neq a}} \beta_{ab}(t_a, t_b) = 0 \quad (a \in A). \quad (15)$$

For $i \in I_a$, set $y_i = \Delta_{a,t_a}(i)$. Equations (12) and (13) give

$$y_i \neq 0 \quad (0 \leq i < n), \quad \sum_{i \in I_a} y_i = 0 \quad (a \in A), \quad (16)$$

while eq. (15) says

$$\sum_{\substack{b \in A \\ b \neq a}} \sum_{\substack{i \in I_a, j \in I_b \\ j < i}} \langle y_i, y_j \rangle = 0 \quad (a \in A). \quad (17)$$

Integration. Since the occurrence sets partition the positions, eq. (16) implies $\sum_{i=0}^{n-1} y_i = 0$. Set $x_{-1} = 0$ and define

$$x_i = \sum_{j=0}^i y_j \quad (0 \leq i < n). \quad (18)$$

Then $x_{n-1} = x_{-1} = 0$, so these colours close around the word, and

$$x_{i-1} + x_i = y_i \neq 0.$$

Thus the two gaps incident with every occurrence have different colours.

Fix a letter a . Its gap incidences carry the $2|I_a|$ colours x_{i-1}, x_i , for $i \in I_a$, and their sum is

$$\sum_{i \in I_a} (x_{i-1} + x_i) = \sum_{i \in I_a} y_i = 0. \quad (19)$$

Using eqs. (3) and (18), then splitting the strict prefix sum according to the occurrence sets, gives

$$\begin{aligned} & \sum_{i \in I_a} (q(x_{i-1}) + q(x_i)) \\ &= \sum_{i \in I_a} q(y_i) + \sum_{\substack{i \in I_a \\ 0 \leq j < i}} \langle y_i, y_j \rangle \\ &= q \left(\sum_{i \in I_a} y_i \right) + \sum_{\substack{b \in A \\ b \neq a}} \sum_{\substack{i \in I_a, j \in I_b \\ j < i}} \langle y_i, y_j \rangle = 0. \end{aligned} \quad (20)$$

The second equality is eq. (4); the last uses eqs. (16) and (17). The family of gap incidence colours at a has even cardinality, zero sum by eq. (19), and zero quadratic sum by eq. (20). By lemma 3.1, every colour occurs in it an even number of times. Since a was arbitrary, the required gap colouring exists. $\square$

Together with the reduction in section 2, this completes the proof of theorem 1.1.

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